towards morphing geodesic graphs on the sphere

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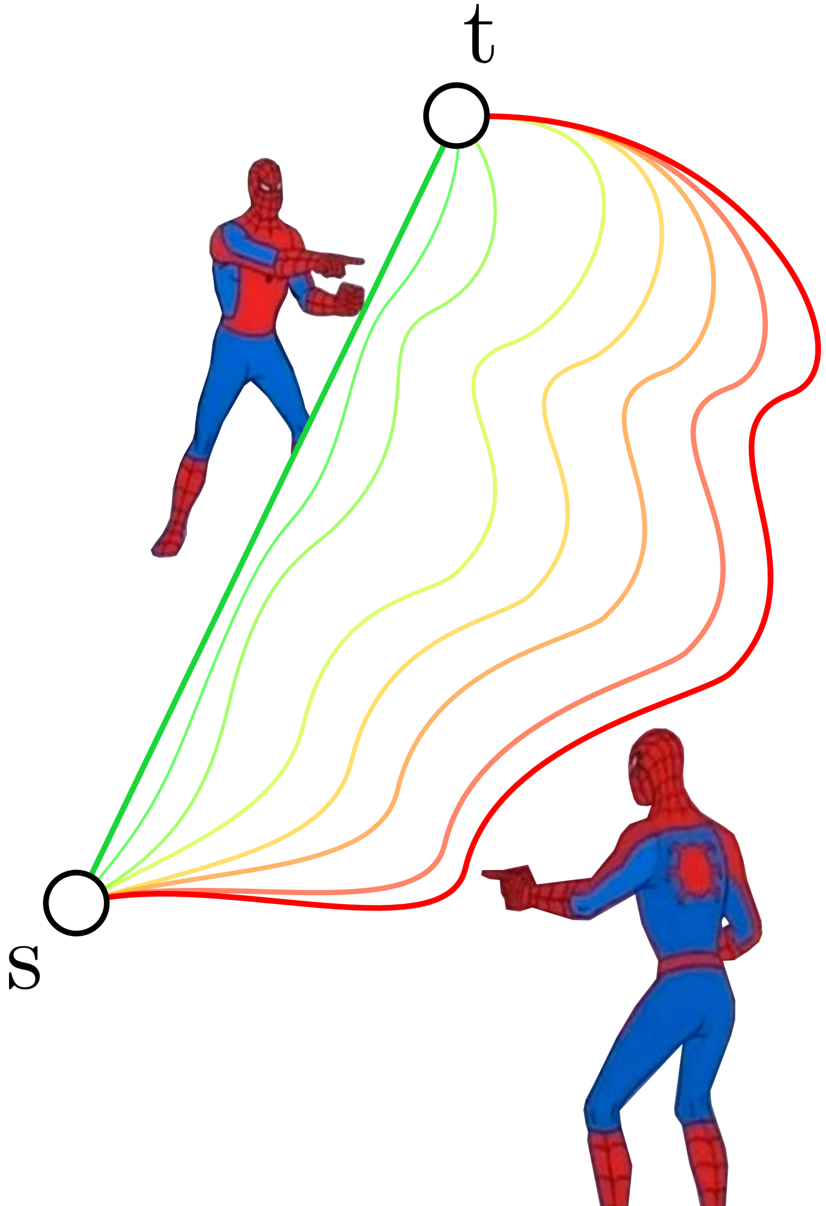
introduction

- **elevator pitch**:
  - CS Theory PhD student @ UIUC
  - Working with advisor Jeff Erickson on computational geometry and topology problems
- **fun fact**:
  - Youngest brother is a student here!

setting the stage...

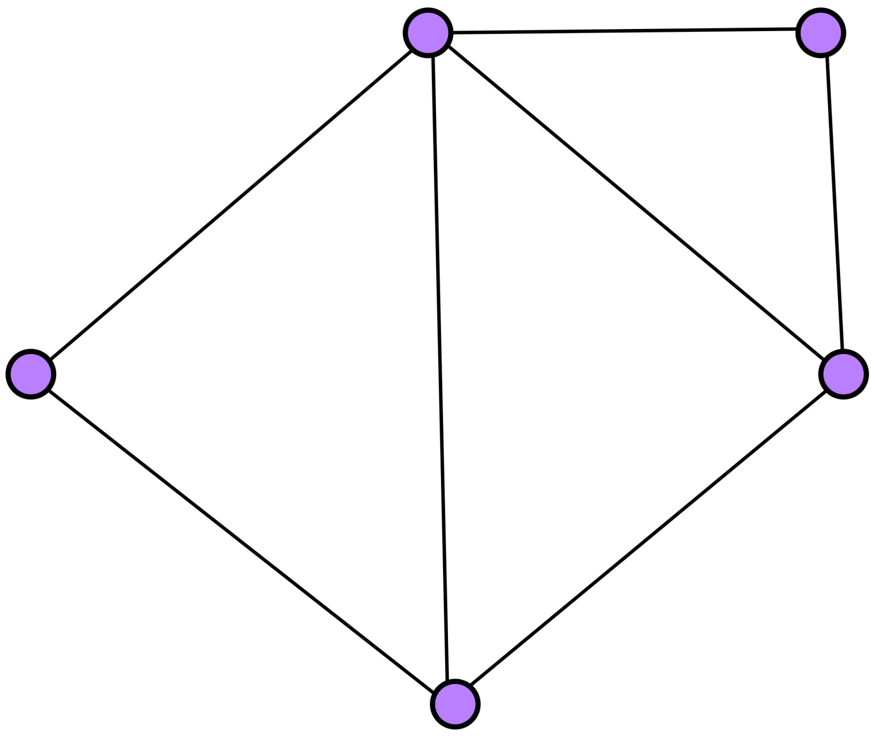
what is a joke about computational topology that I can say to start a technical talk?

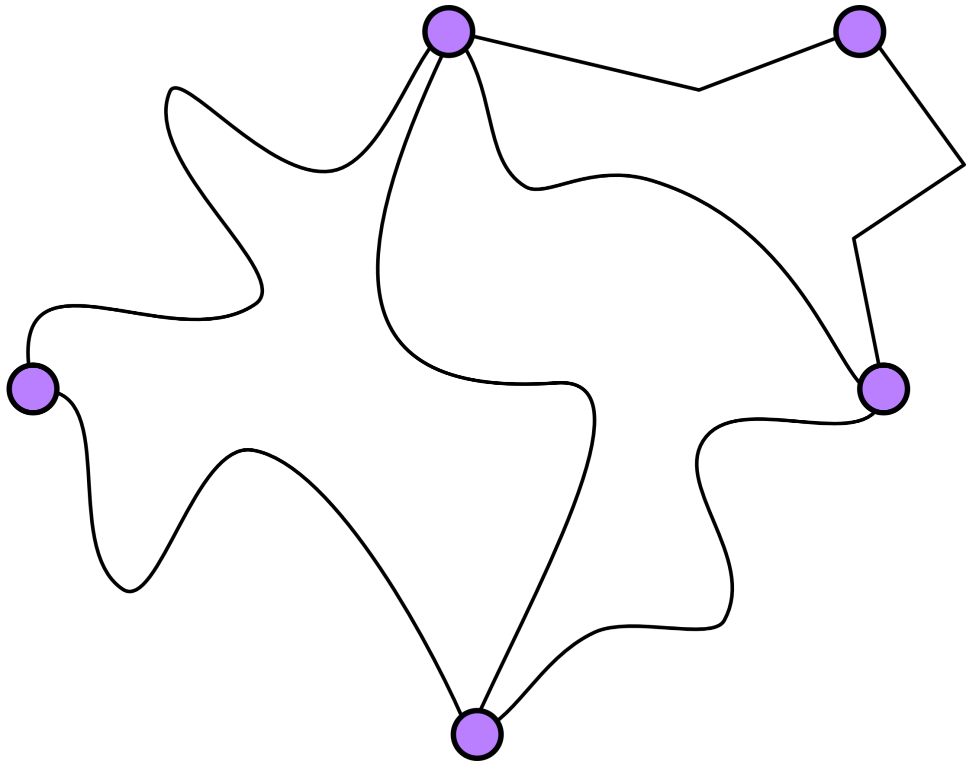
why did the computational topologist get lost on the way to the conference?
because they couldn't tell the difference between a shortcut and a homotopy!



background

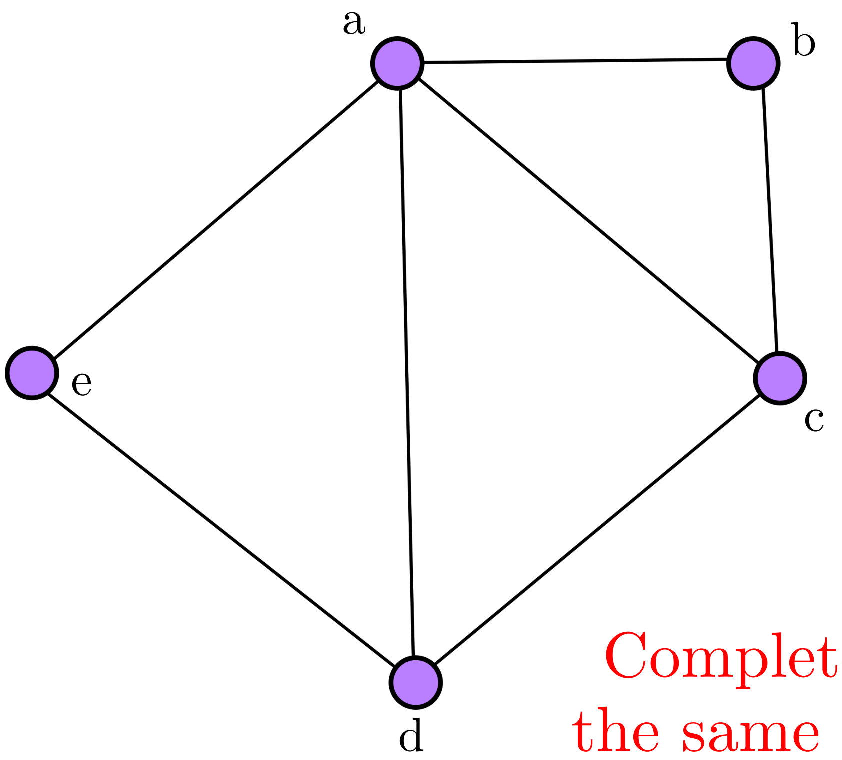
- Isomorphic embeddings



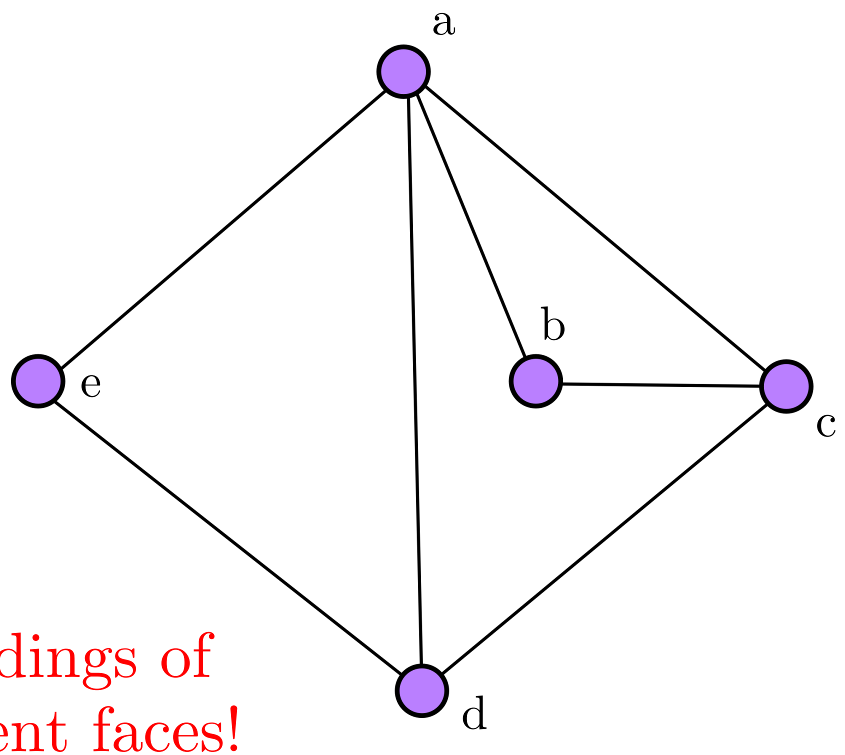


background

- Isomorphic embeddings must have the same faces

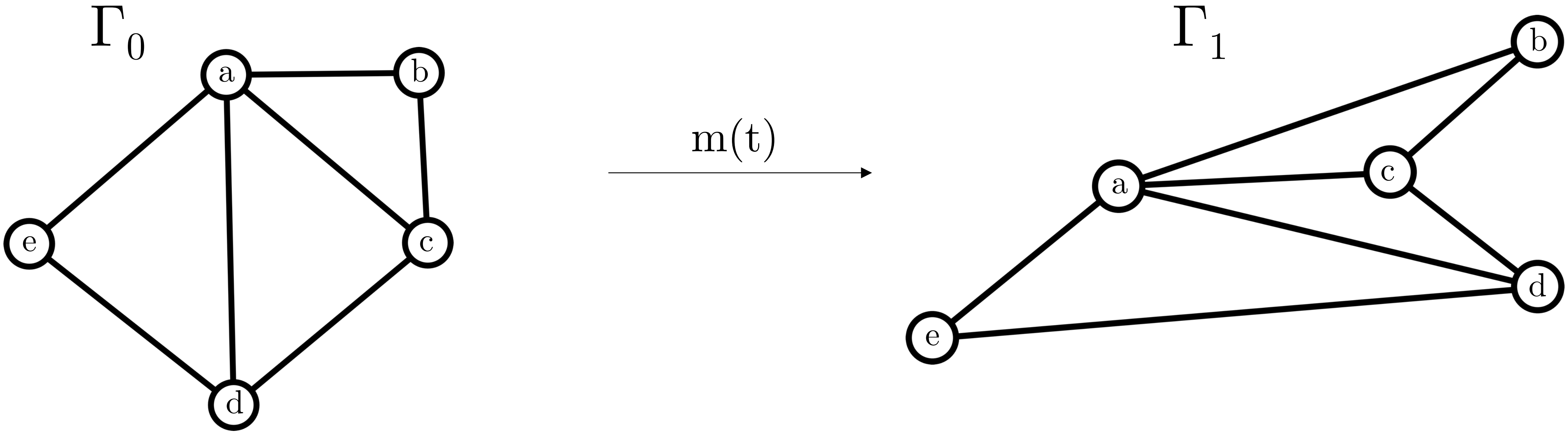


Completely different embeddings of
the same graph due to different faces!



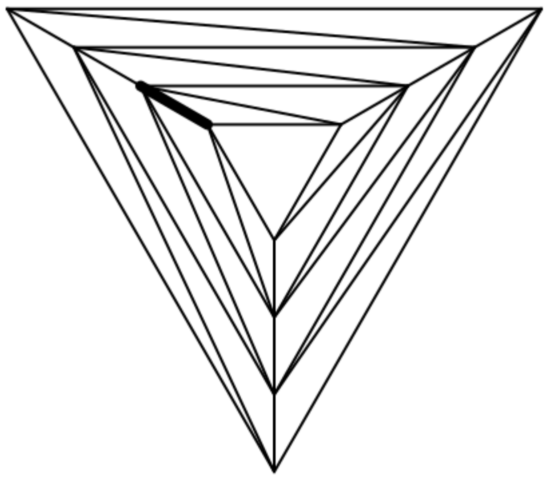
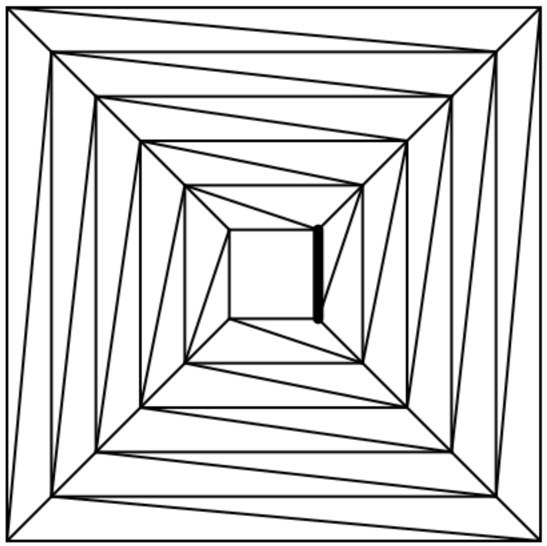
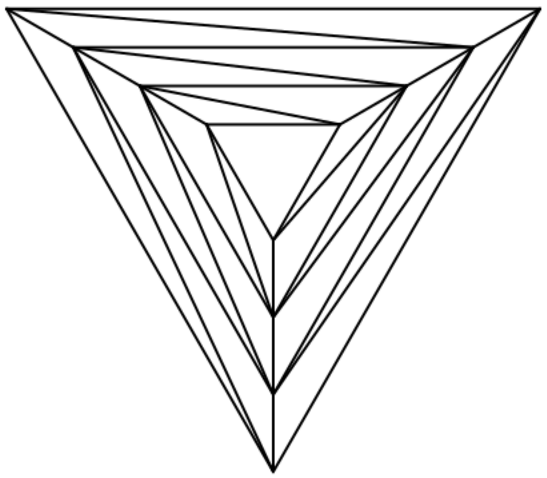
background

- Morph
  - Continuous function m such that m(t) is an injective embedding of graph G for every t ∈[0, 1], where $m(0) = \Gamma_0$ and $m(1) = \Gamma_1$



background

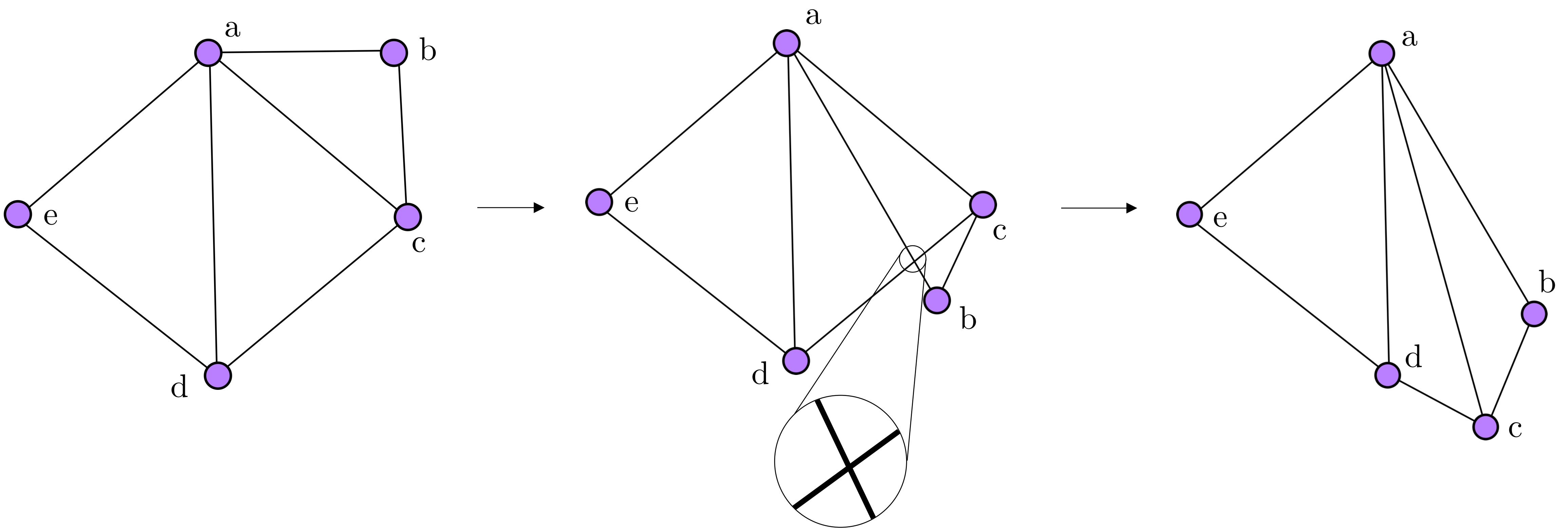
- Example morphs

Morph examples provided by Jeff Erickson

background

- **Invalid morph**
  - Suppose morph vertex by vertex



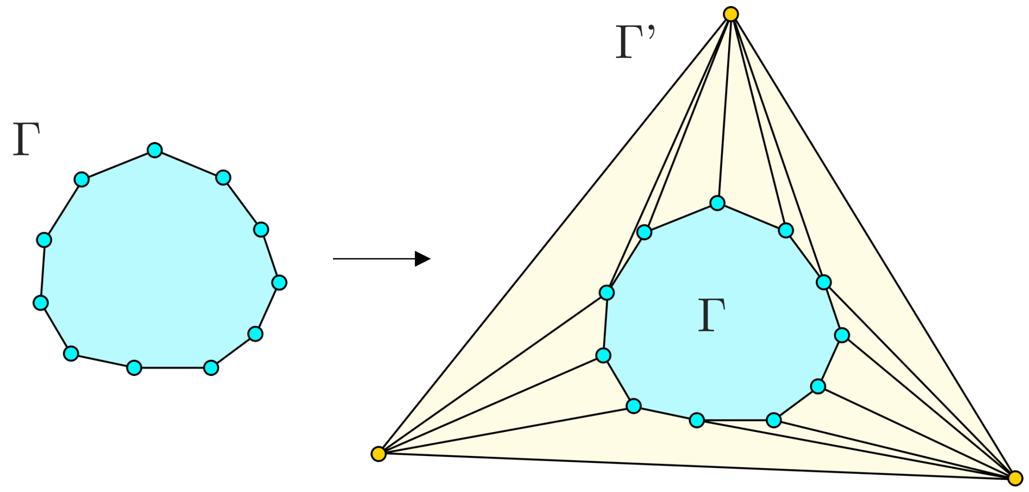
motivation

- Practical uses of morphing between spherical embeddings
  - Smoothly morphing meshes on the sphere in graphics context

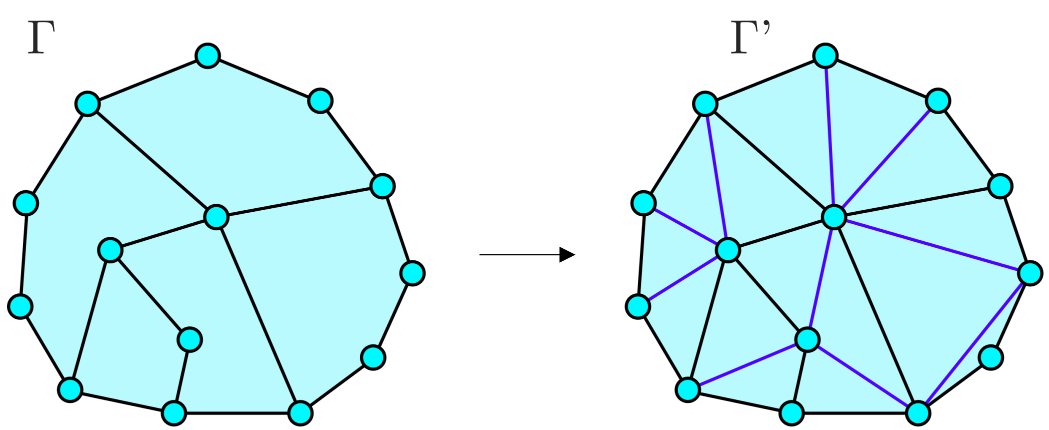
There exist efficient methods that seem to do well in practice but they lack theoretical guarantees that the embedding will remain injective through the transformation

planar morphing

Theorem (Cairns, 1944): Given two isomorphic straight line triangulations Γ_0 and Γ_1 in the plane with a triangle outer face, there exists a morph from Γ_0 to Γ_1 . If the embeddings are on n vertices, the morph can be constructed in $O(2^n)$ morphing steps.



Handling convex outer faces

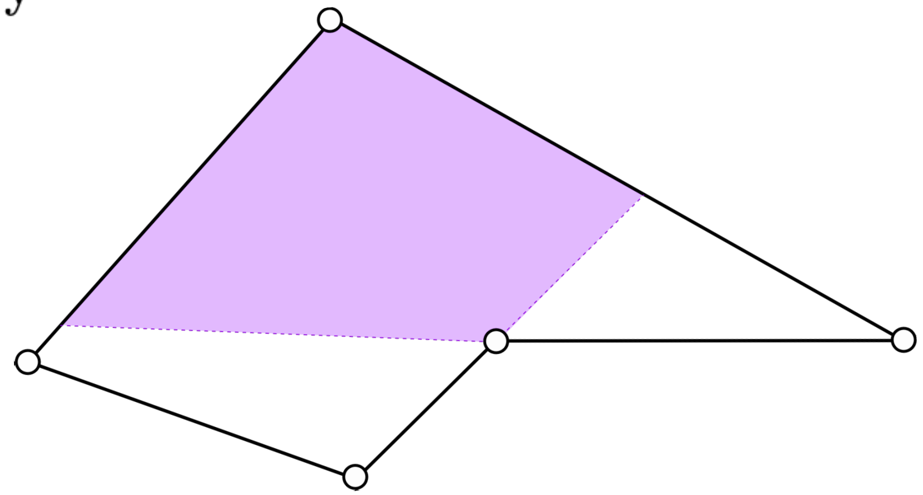


Handling non-triangulations

planar morphing

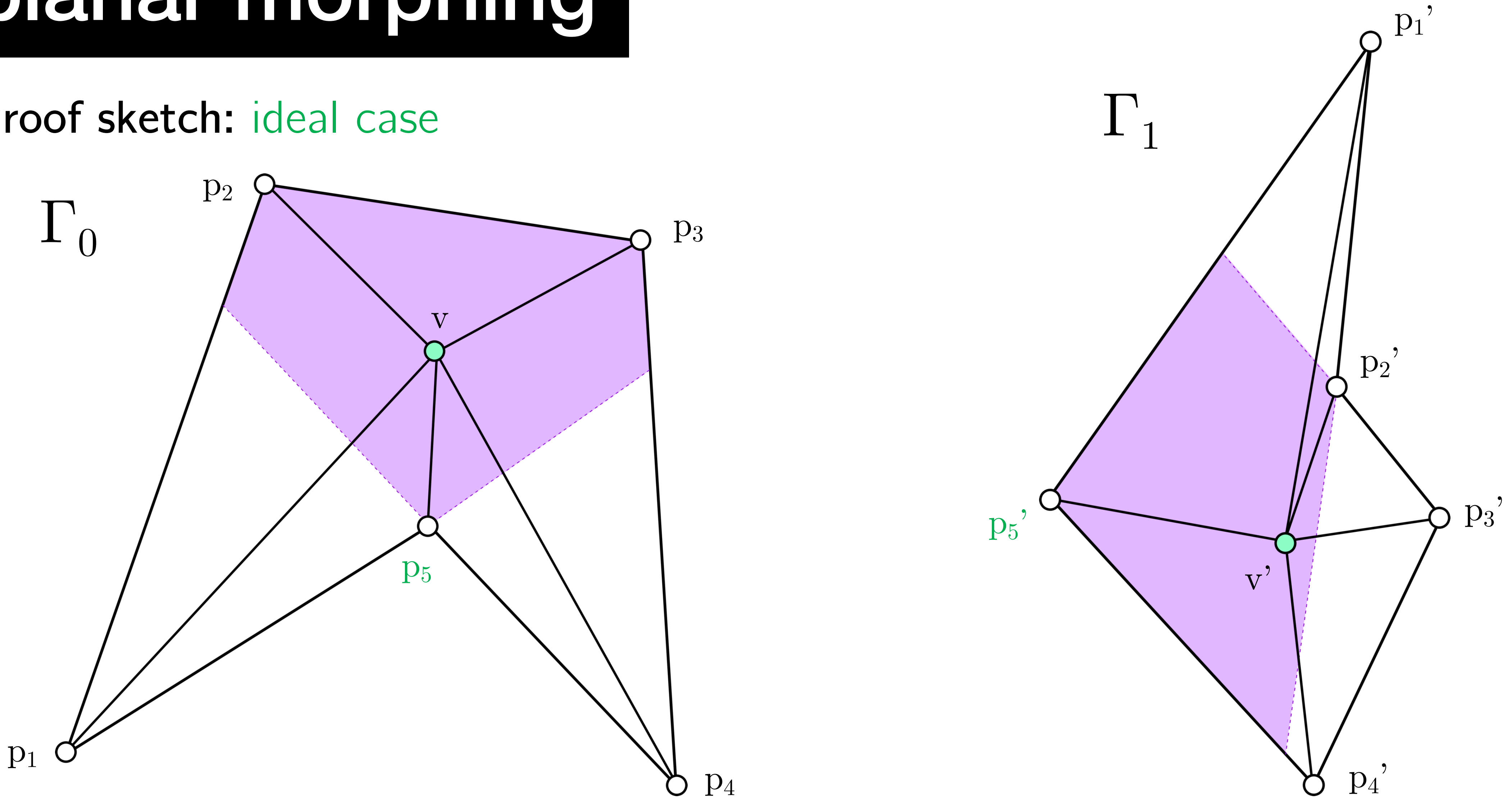
Proof sketch

- Assume we can morph between two isomorphic embeddings Γ and Γ' on at most $(n-1)$ vertices.
- Since Γ_0 and Γ_1 are planar, there exists a vertex with degree at most 5
- Can show that for any pentagon, its kernel is non-empty and contains at least one vertex on the boundary



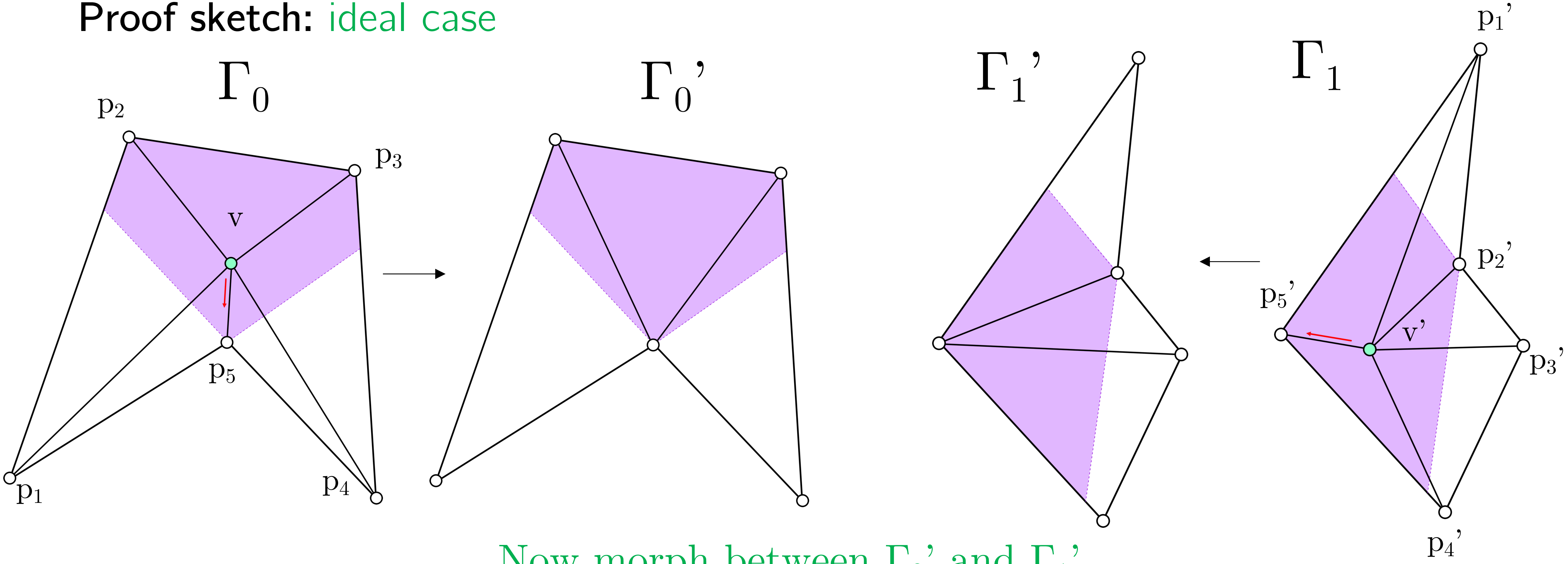
planar morphing

Proof sketch: ideal case



planar morphing

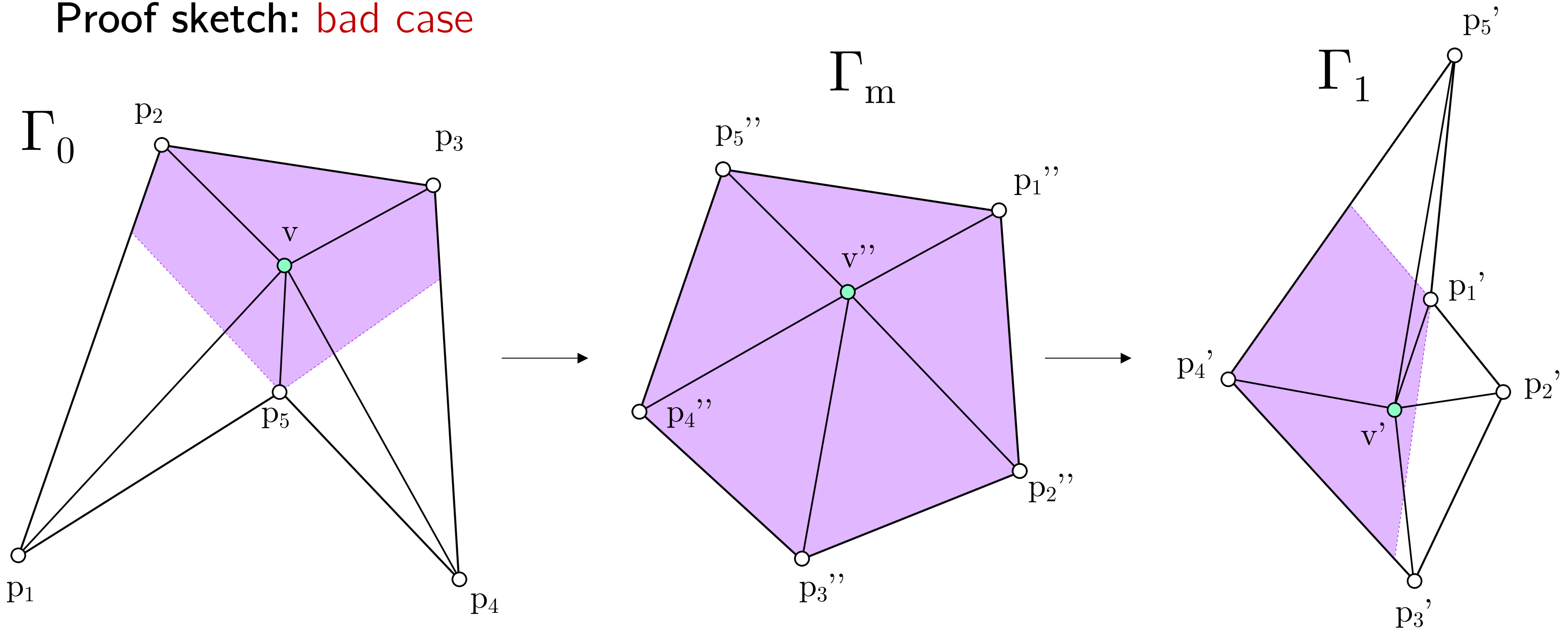
Proof sketch: ideal case



Now morph between Γ_0' and Γ_1'
since they each have (n-1) vertices!

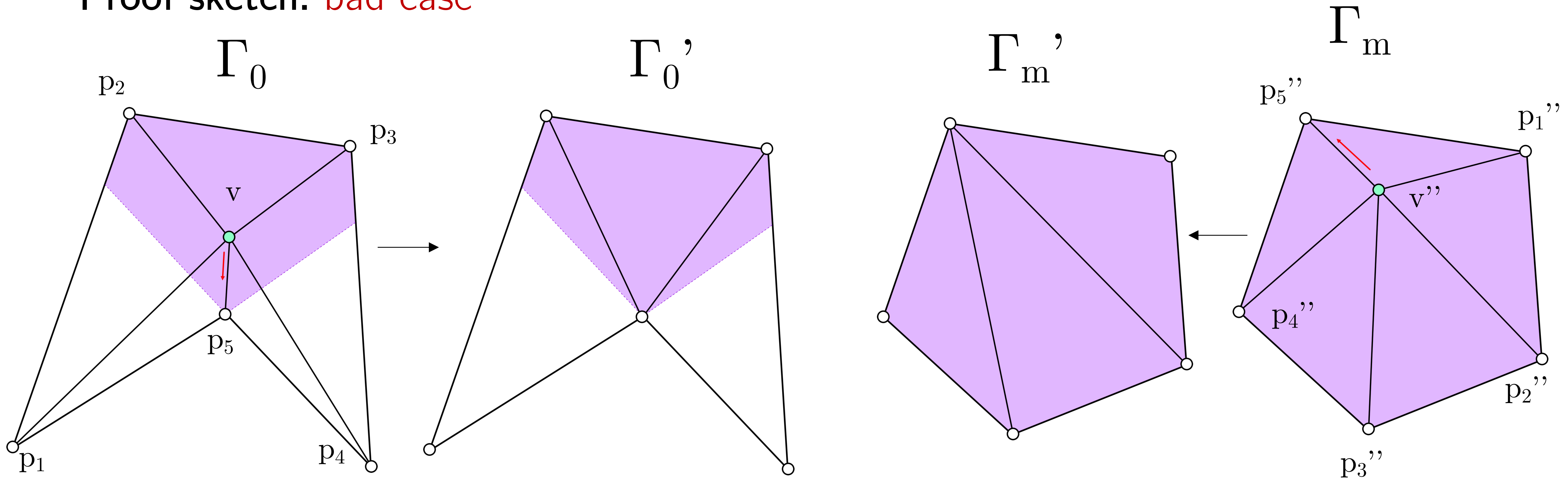
planar morphing

Proof sketch: bad case



planar morphing

Proof sketch: bad case



Now morph between Γ_0' and Γ_m'. Need to do similar work to morph from Γ_m to Γ_1.

planar morphing

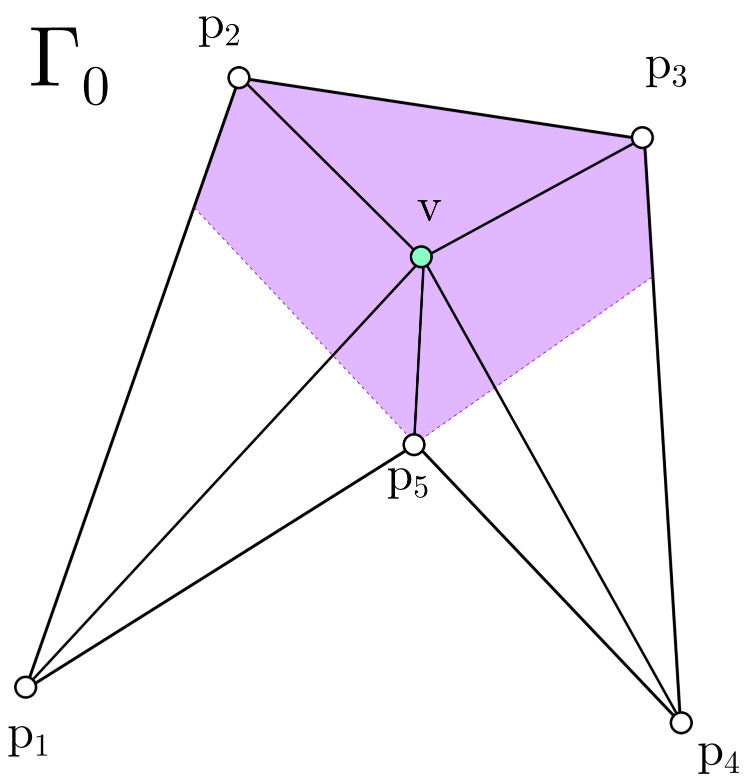
Proof sketch: runtime analysis

- $T(n)$ is the number of morphing steps needed to morph between embeddings Γ_0 and Γ_1 on n vertices
- Then $T(n) \leq 2T(n-1) + O(1)$ with $T(3) = O(1)$
- Verify by induction that $T(n) = O(2^n)$

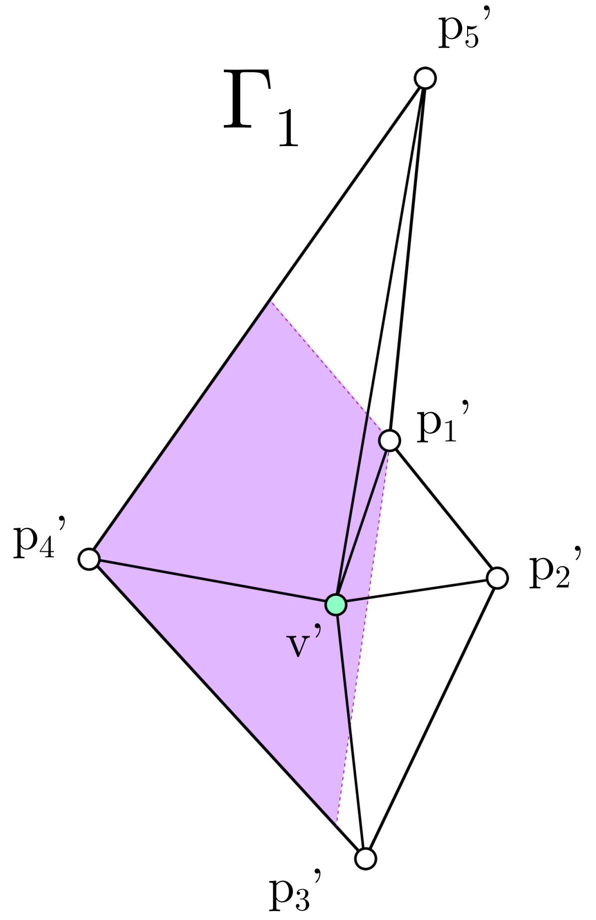
Want something with a polynomial
number of morphing steps

planar morphing

Making things more efficient: Revisiting bad case

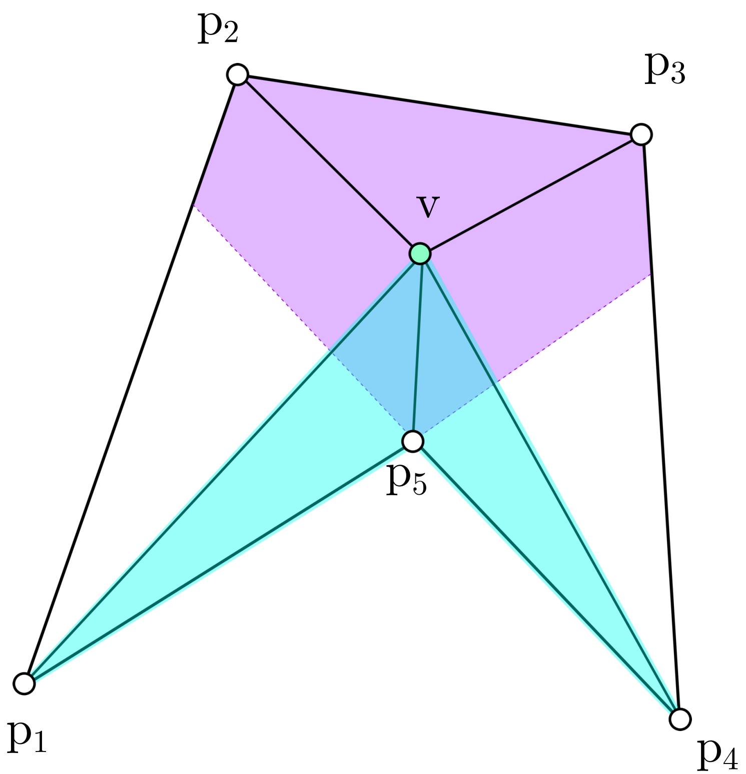


Want shared vertex on boundary in kernel



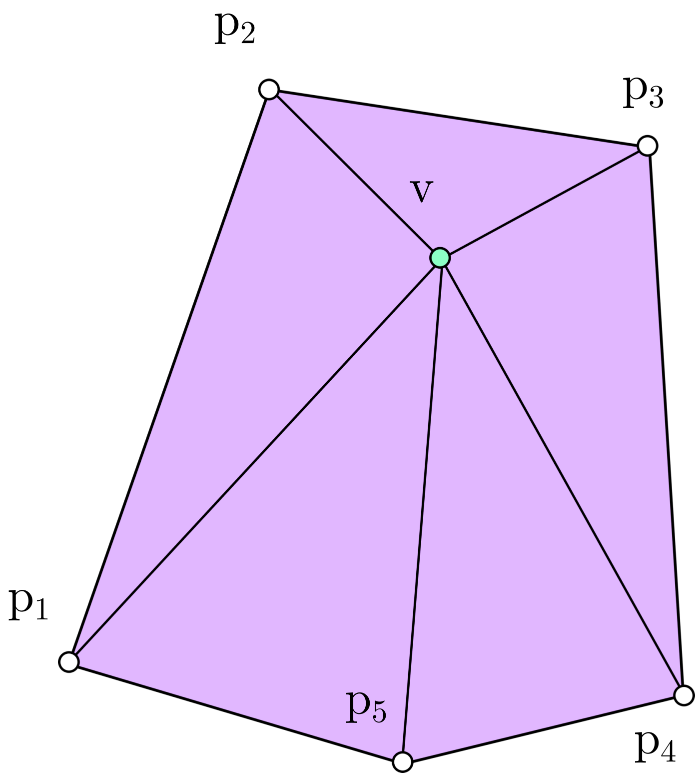
planar morphing

Convexifying quadrilaterals



Convexify Quad

May need to do this a constant number of times in the pentagon



planar morphing

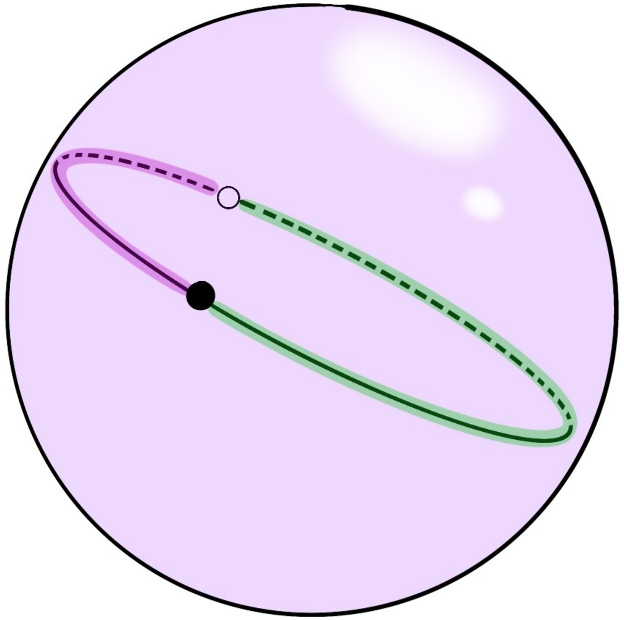
Theorem (Alamdari et al., 2017): Given two isomorphic straight line embeddings Γ_0 and Γ_1 in the plane with a shared outer face, there exists a morph from Γ_0 to Γ_1 . If the embeddings are on n vertices, the morph can be constructed in $O(n)$ unidirectional morphing steps.

Lemma (Alamdari et al., 2017): Given a straight line embedding Γ with nonconvex quadrilateral abcd with no vertex inside it and no external chords, abcd can be made convex by $O(1)$ unidirectional morphing steps.

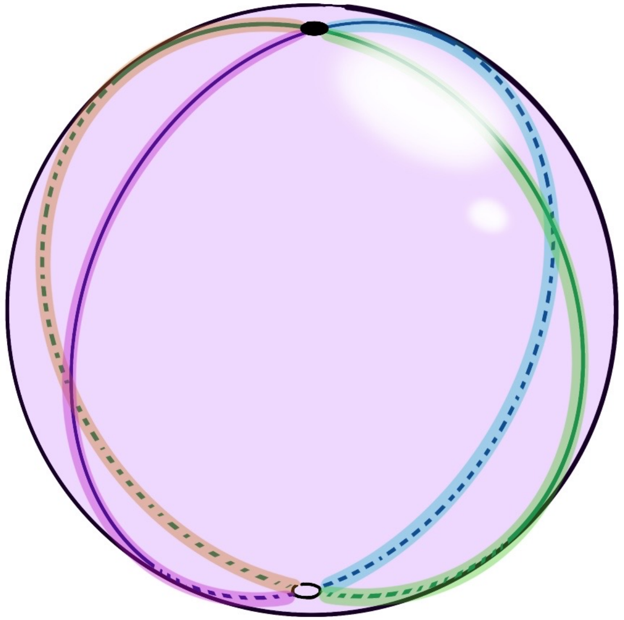
Built off Cairns' work but uses quadrilateral convexification to ensure they always end up in the good case when doing the recurrence

spherical morphing

- **Problem**: Given two isomorphic spherical embeddings Γ_0 and Γ_1 with edges drawn as geodesics, construct a morph from Γ_0 to Γ_1



Geodesics for non-antipodal vertices



Geodesics for antipodal vertices

spherical morphing

- Important related results:
  - (Steinitz, 1922): Proved there is a bijection between planar 3-connected graphs and convex polyhedra; also gave constructive proof that one can morph between isomorphic convex polyhedra; leads to special case morphing algo with poly(n) morphing steps

spherical morphing

- **Important related results**:
  - (Awartani, Henderson, 1987): Proved the following
    1. Can morph between two embeddings where every fixed vertex has the same longitude in both embeddings; leads to algo with O(1) morphing steps
    2. Under special conditions, a spherical triangulation with shortest geodesics can be efficiently morphed to the southern hemisphere; leads to special case morphing algo with O(n) morphing steps

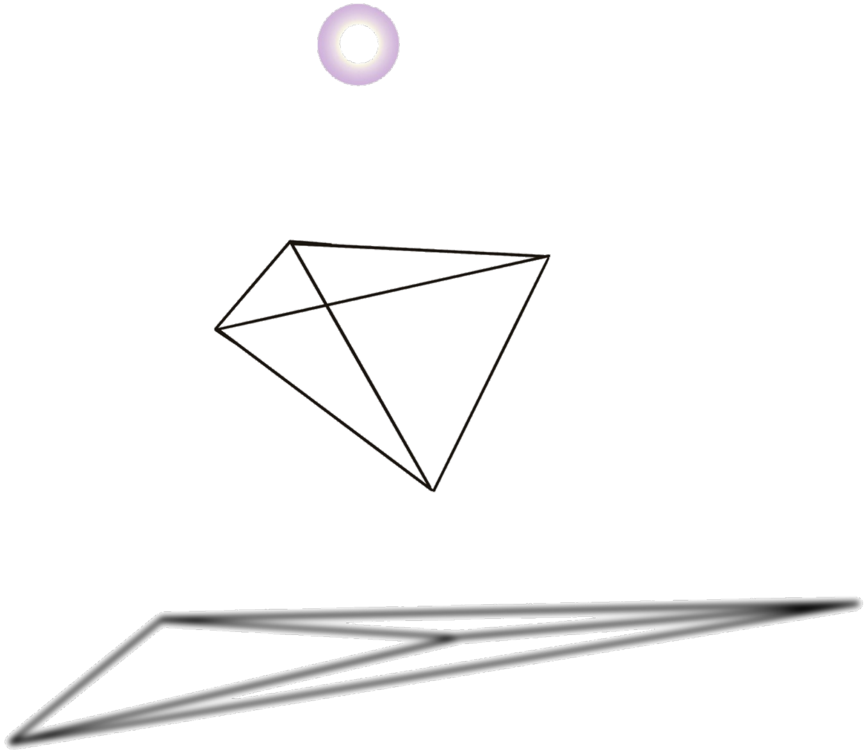
spherical morphing

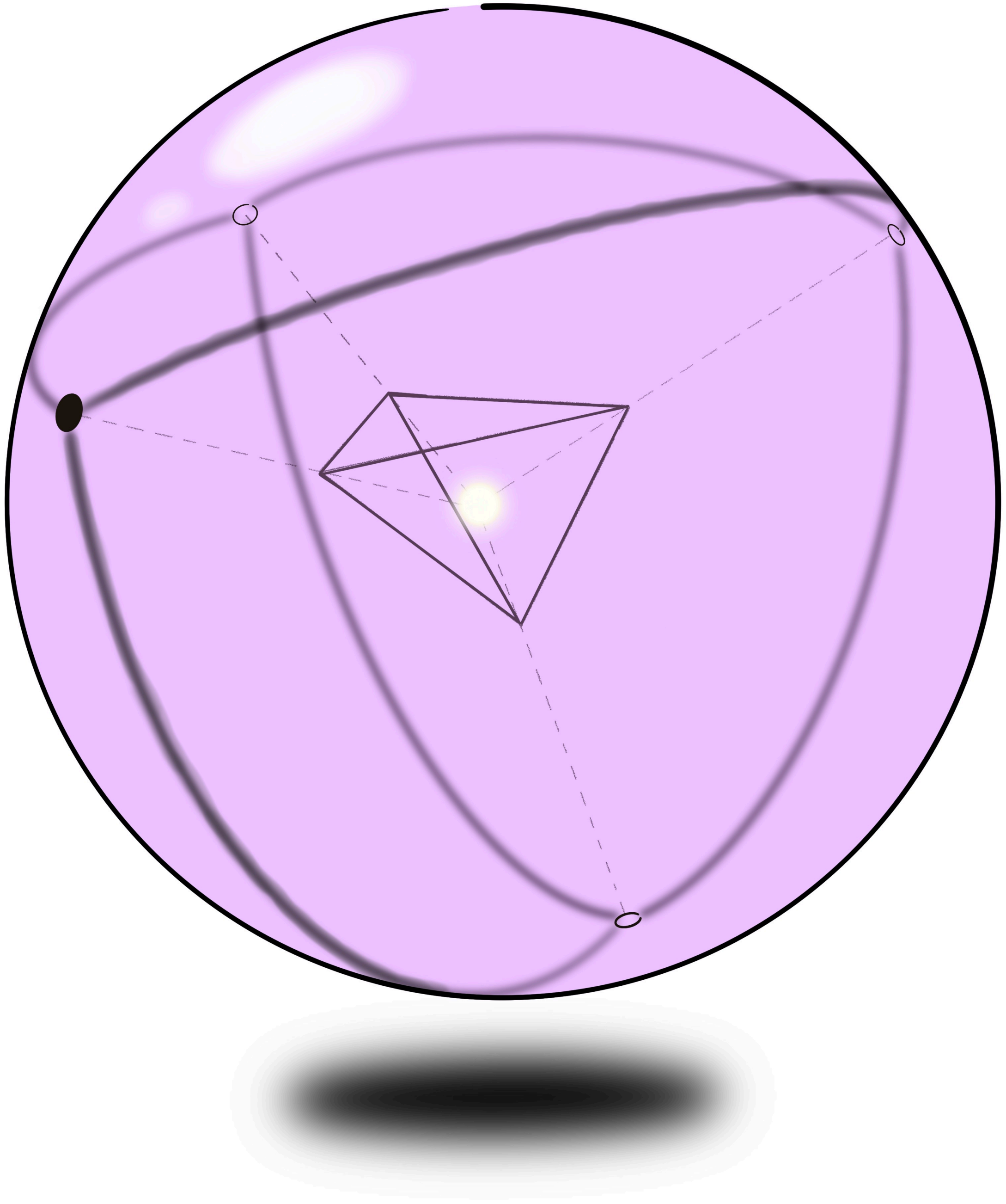
- **Important related results**:
  - (Cairns, 1944): Proved constructively there exists a morph between spherical triangulations with shortest geodesics; leads to algo with $O(2^n)$ morphing steps

Want poly(n) morphing steps

spherical morphing

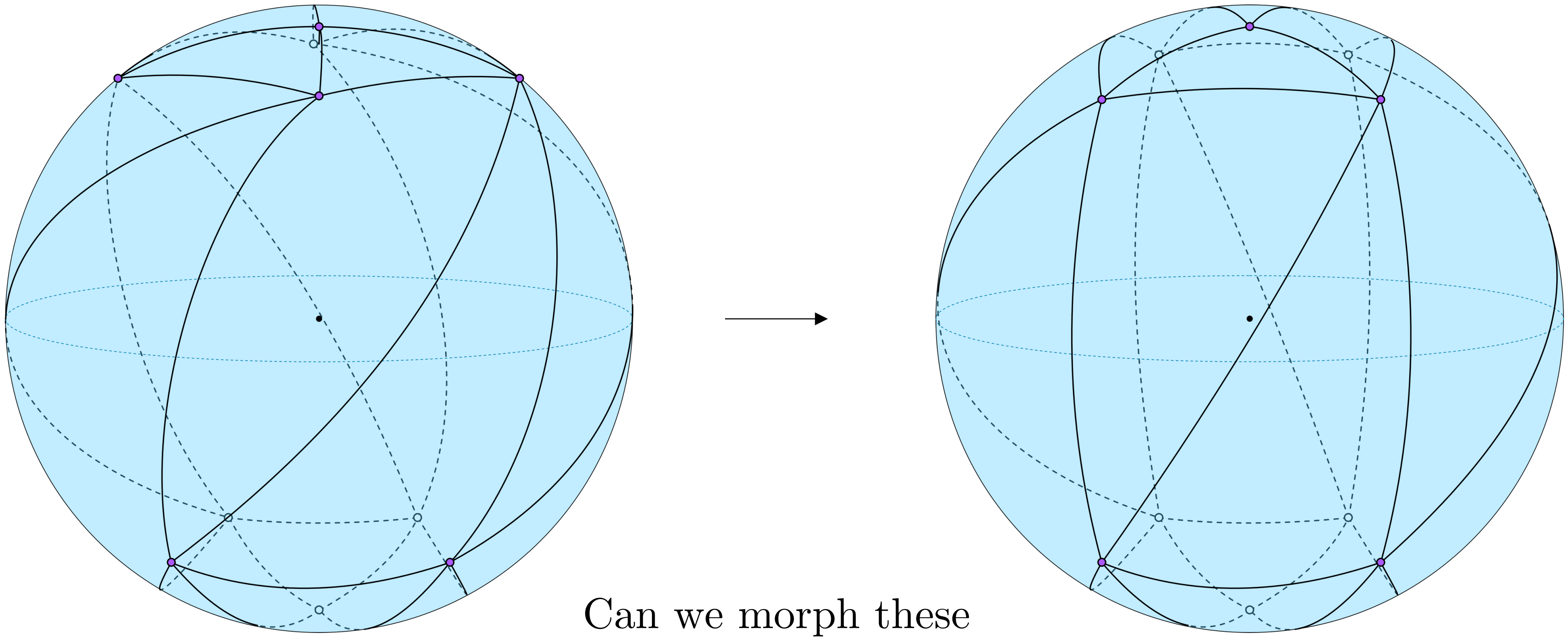
- Theorem (Steinitz, 1922):





spherical morphing

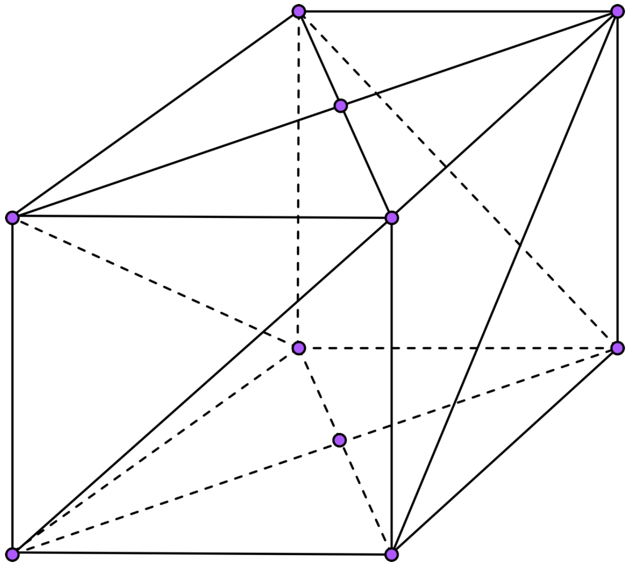
- Theorem (Steinitz, 1922):



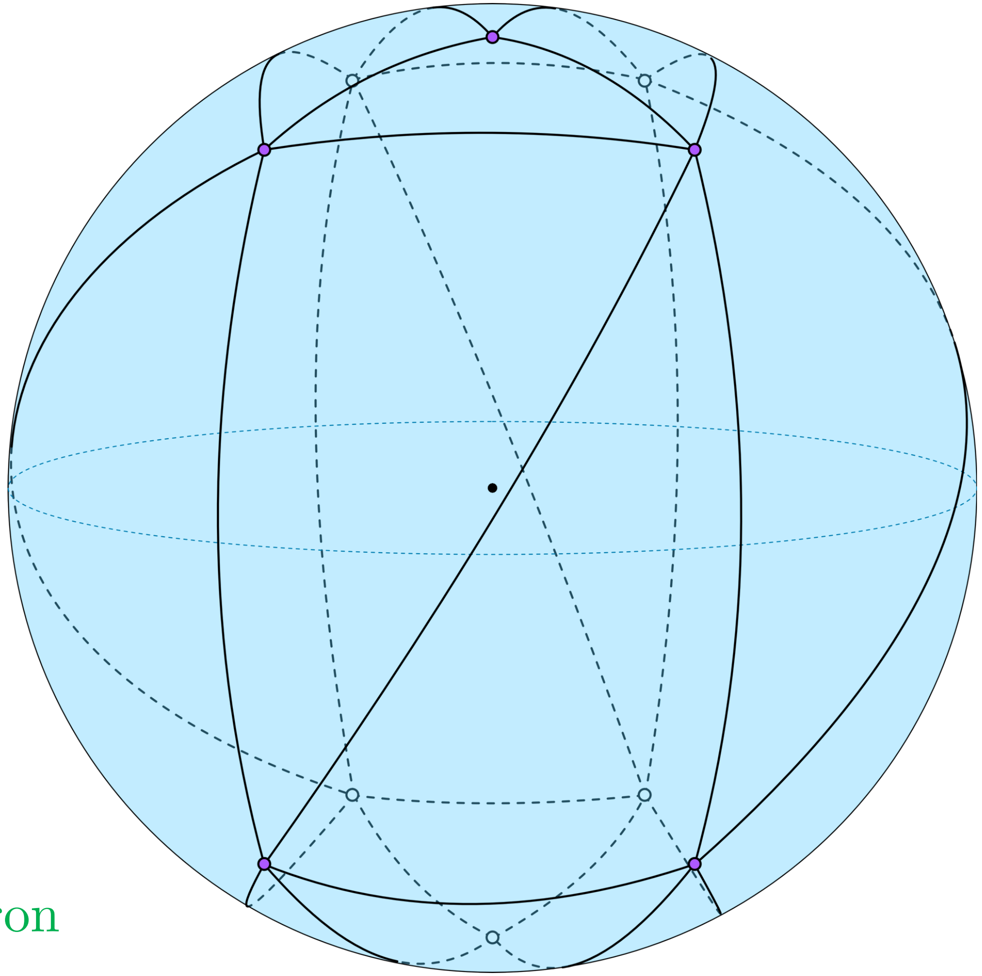
Can we morph these
embeddings with Steinitz?

spherical morphing

- Theorem (Steinitz, 1922):

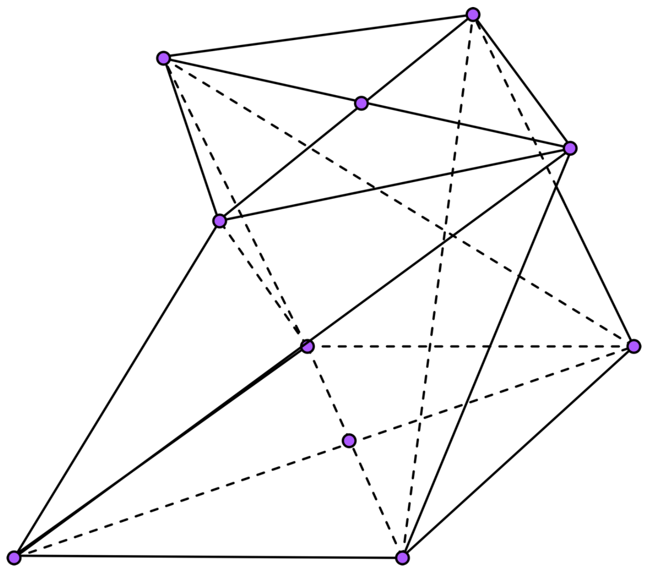


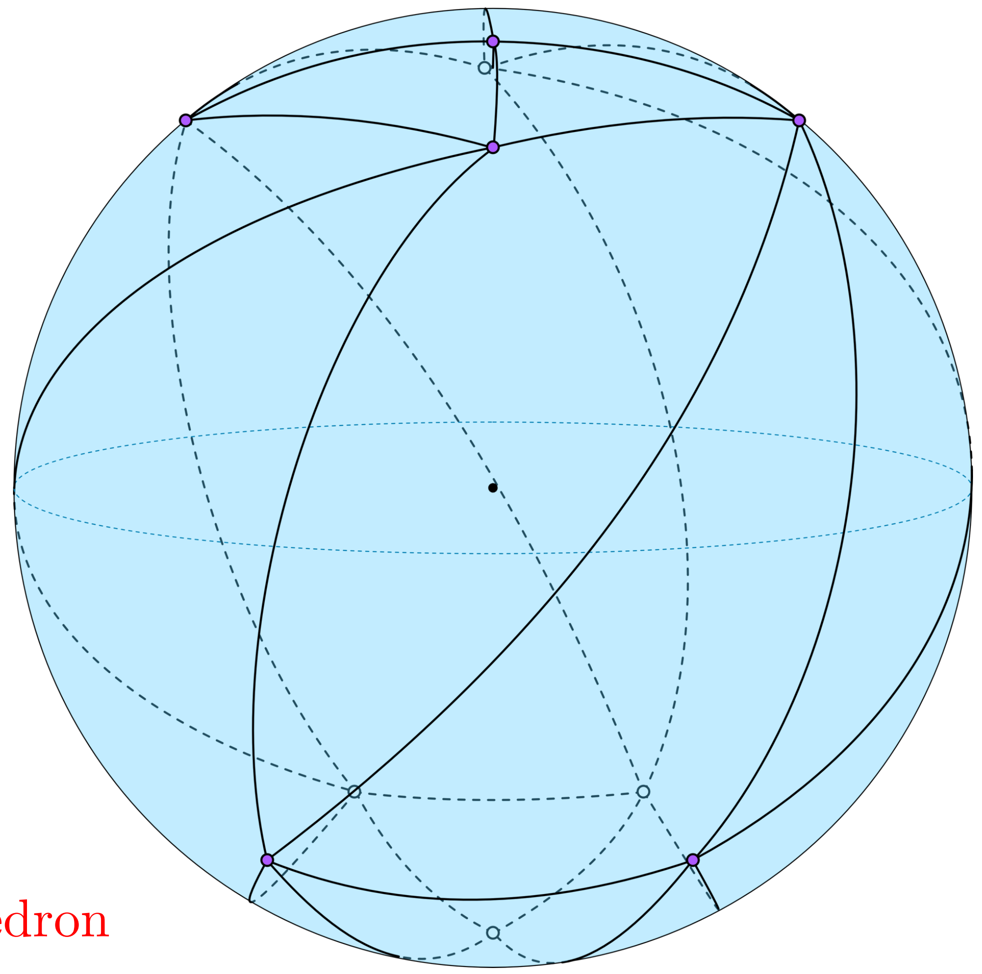
There exists convex polyhedron
for spherical embedding



spherical morphing

- Theorem (Steinitz, 1922):

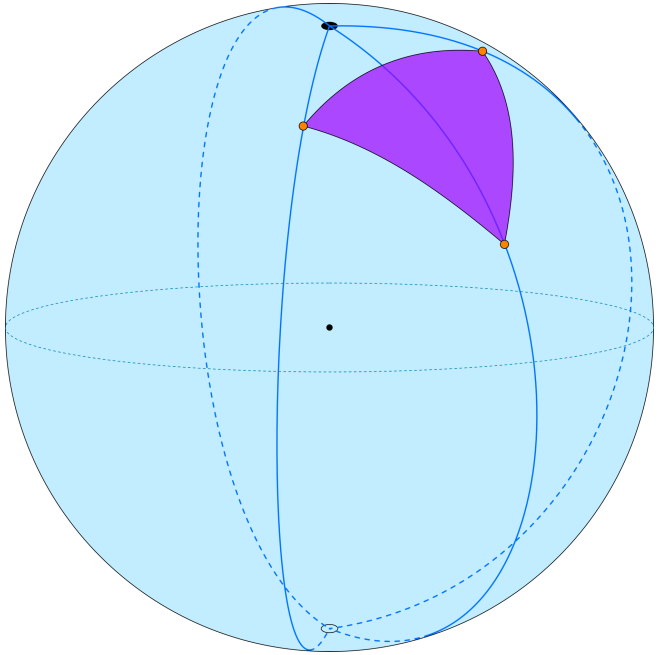


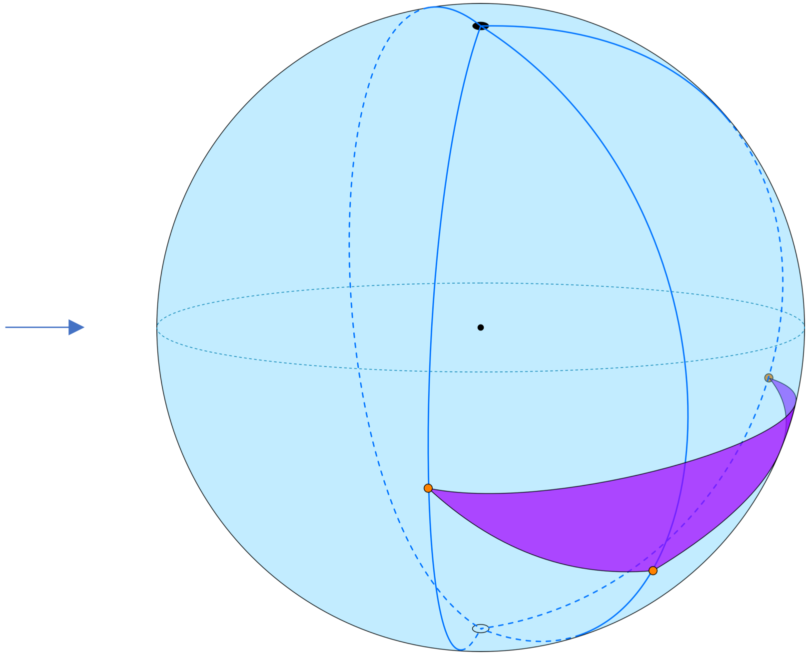


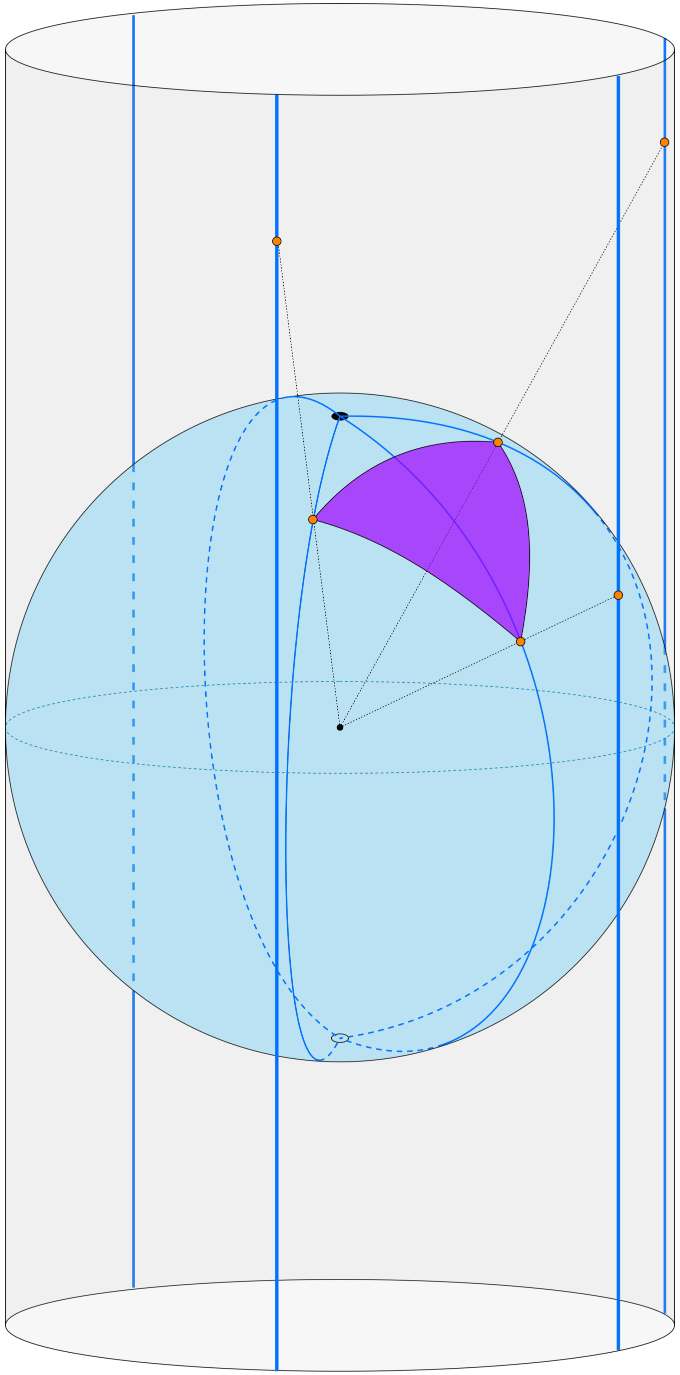
There does not exist convex polyhedron
for this spherical embedding

spherical morphing

- **Lemma (Awartani, Henderson, 1987)**: Fixed longitude morphs

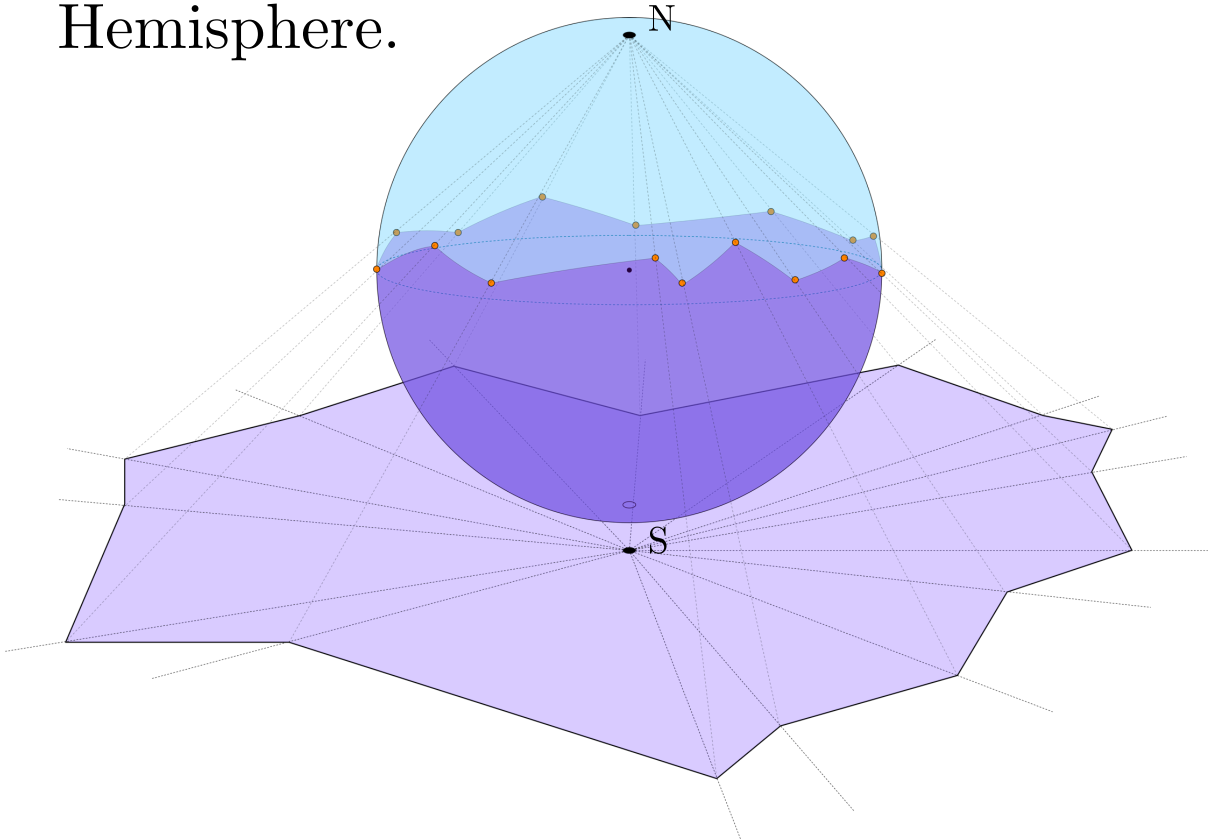


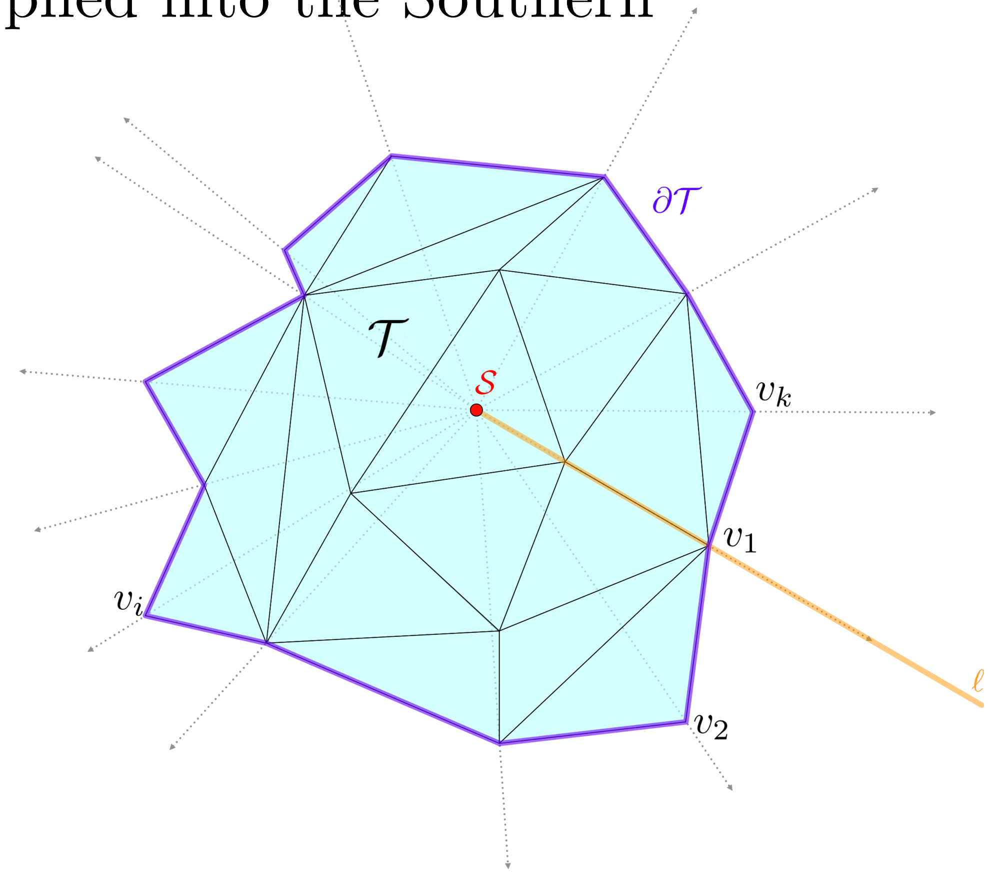




spherical morphing

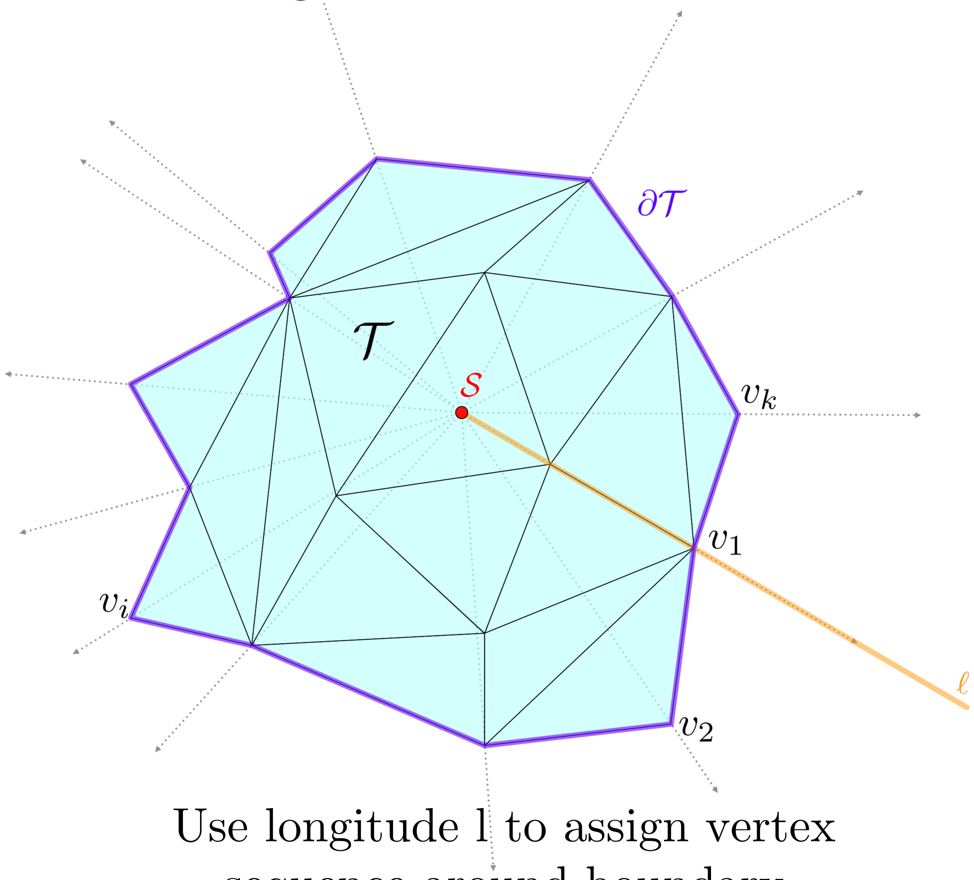
- **Theorem (Awartani, Henderson, 1987)**: If a triangulation K has a longitude l such that no edge intersects l at exactly 1 interior point, then the triangulation T := K\star(N) can be morphed into the Southern Hemisphere.



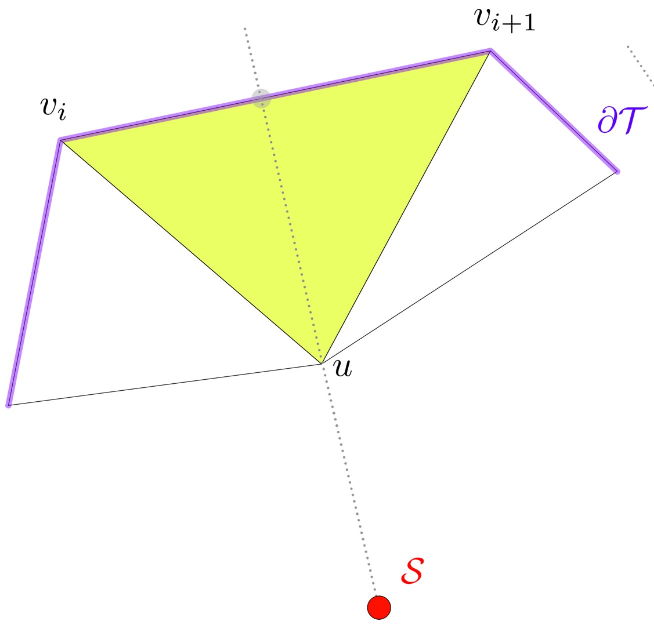


spherical morphing

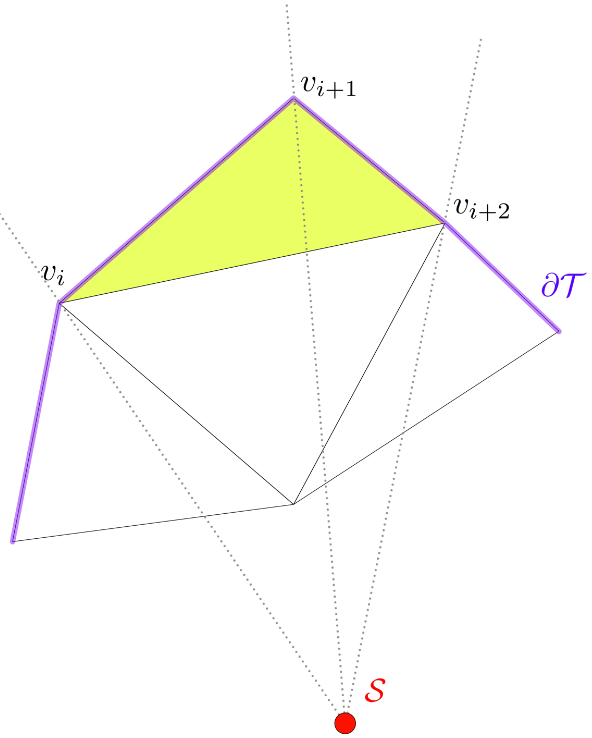
- Proof Sketch:



Use longitude l to assign vertex sequence around boundary



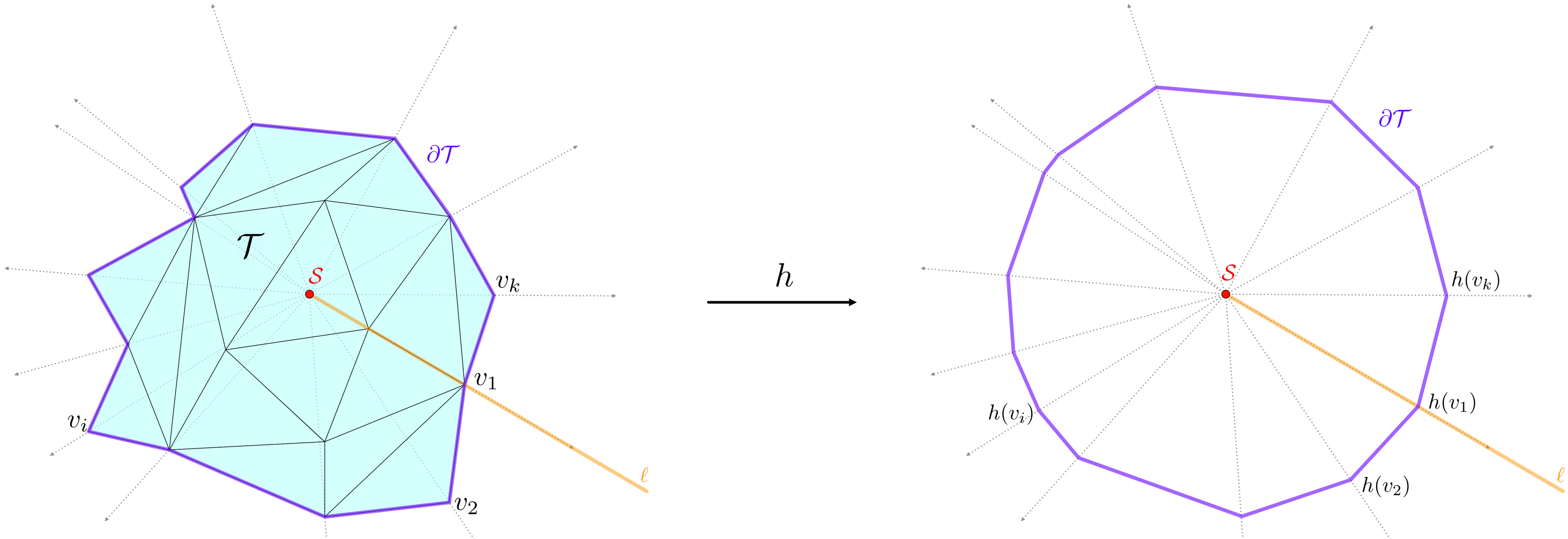
Normal triangle



Properly spanning triangle

spherical morphing

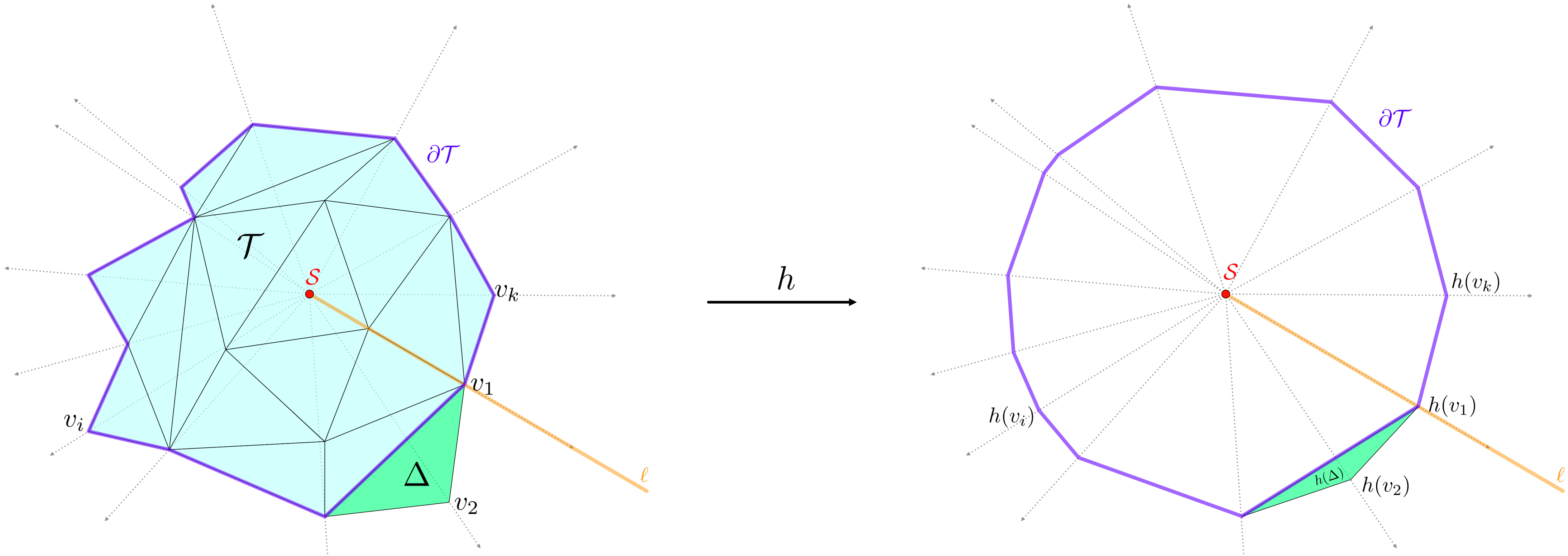
- Proof Sketch:



Given there are n triangles in our triangulation, initialize new embedding with convex boundary in Southern Hemisphere and assume we can form a new embedding within a specified convex boundary for all embeddings with at most (n-1) triangles

spherical morphing

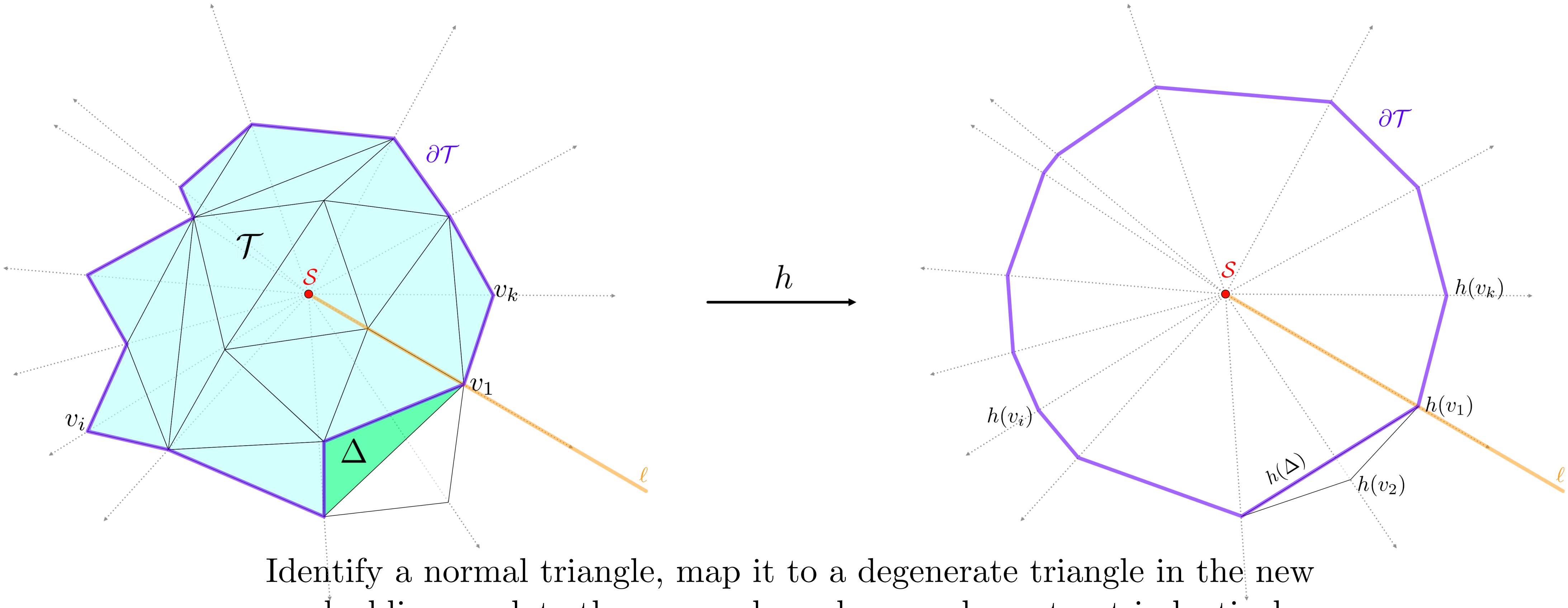
- **Proof Sketch**: Properly spanning triangle case



Identify a properly spanning triangle, map it to its exact corresponding triangle in the new embedding, update the convex boundary, and construct inductively

spherical morphing

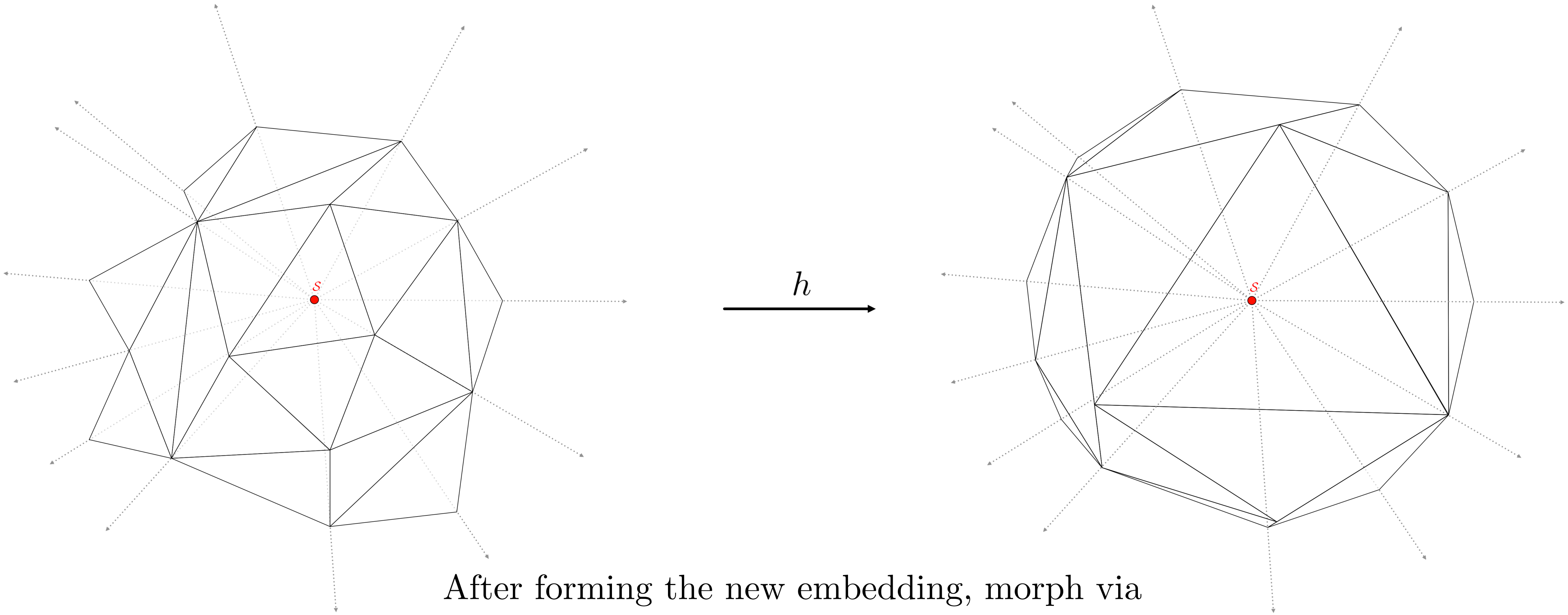
- **Proof Sketch**: Normal triangle case



Identify a normal triangle, map it to a degenerate triangle in the new embedding, update the convex boundary, and construct inductively

spherical morphing

- Proof Sketch:



After forming the new embedding, morph via the fixed longitude morph in 1 morphing step

spherical morphing

- **Comments**: If two input embeddings with the same fixed north pole satisfy the longitude condition, then we can morph between the embeddings in $O(n)$ morphing steps

- **Problem**: Given two spherical triangulations Γ_0 and Γ_1 with short geodesics, can we fix some vertex as the North Pole and morph both embeddings such that the longitude condition in the theorem is true?

If we can solve this, we can morph both embeddings to the Southern Hemisphere and use a central projection to project them to the plane, allowing us to use planar morphing tools

spherical morphing

- **Cairns (again)**: Using a similar approach to the planar case, Cairns proved that we can morph between two triangulation spherical embeddings, so long as their edges are shortest geodesics, in $O(2^n)$ morphing steps

Want poly(n) morphing steps

- **Problem**: Can the quadrilateral convexification problem be solved in poly(n) morphing steps on the sphere?

If one solves the above problem, Cairns' work can be made efficient similarly to the work done by Alamdari et al.

Henderson et al. gives a fixed longitude morph, potentially useful subroutine to solve this problem

spherical morphing

- General Case Comments
  - General case allows for long geodesic edges
  - By the intermediate value theorem, we know if an edge is short in one embedding and long in the other, we will hit a moment where the vertices for that edge are antipodal
    - This implies we need to carry information about which geodesic we are using since antipodal vertices have infinitely many geodesics

questions?

backup ...

planar morphing

- Mention Floater-Gotsman

spherical morphing

- Floater-Gotsman styled approach using Colin de Verdière matrices
  - ideas and challenges

background

- Rotation system: $(D, \text{rev}, \text{succ})$